Simple jumping process with memory: Transport equation and diffusion

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We present a stochastic jumping process, defined in terms of jump-size probability density and jumping rate, which is a generalization of the well-known kangaroo process. The definition takes into account two process values: after and before the jump. Therefore, the process is able to preserve memory about its previous values. It possesses a simple stationary limit. Its master equation is interpreted as the kinetic equation with variable collision rate. The process can be easily applied to model systems which relax to distributions other than Maxwellian. The case of a constant jumping rate corresponds to the diffusion process, either normal or ballistic.

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The kangaroo process (KP) [1] is a special kind of jumping process, defined by means of two quantities: the probability density $Q(x)$—where x is registered just after a jump—and the jump rate $\nu(x)$. Therefore the kernel of the corresponding master equation can be factorized with respect to process values before and after a jump; i.e., the subsequent jumps are independent. That feature allows us to solve the master equation relatively easy [2]. Due to its simplicity, the KP is very useful in applications. It was originally invented to solve the problem of Stark broadening in the framework of nonlinear spectroscopy [3], then introduced as a description of turbulent transport in fluids [4]. Since the KP can be so constructed to possess a covariance which is given *a priori*, it serves as a model of colored noises [5–8]. It has also been used as a collision kernel in the linear Boltzmann equation [9].

The independence of subsequent jumps is a strong requirement and rarely satisfied. In this paper we consider a generalized version of the kangaroo process (GKP). The modification consists in allowing the distribution of the process value after jump (x) to depend on the value before the jump (x'): $Q(x)\rightarrow Q(x',x)$. The normalization condition is of the form $\int Q(x',x)dx=1$.

GKP is then a stepwise constant Markov process defined for infinitesimal time intervals Δt by the following stationary transition probability:

$$p_{tr}(x,\Delta t|x',0)=\{1-\nu(x')\Delta t\}\delta(x'-x)+\nu(x')\Delta t Q(x',x). \tag{1}$$

The master equation derived from the above transition probability is

$$\frac{\partial}{\partial t}p(x,t)=-\nu(x)p(x,t)+\int Q(x',x)\nu(x')p(x',t)dx'. \tag{2}$$

From now on, we restrict our analysis to distributions which are determined on the interval $x\in(-\infty,\infty)$ and which depend only on the difference of their arguments: $Q(x',x)=Q(x-x')$. In the physically most interesting case $Q(\xi)$ has a maximum at $\xi=0$ and then the above assumption means that a small jump is easier than a large one.

From Eq. (2) we get the equation for stationary distribution $P(x)$:

$$P(x)\nu(x)=\int Q(x-x')P(x')\nu(x')dx'. \tag{3}$$

Expanding $P(x)\nu(x)$ in a power series, one can show that for any analytic, positive $P(x)\nu(x)$ and normalized $Q(\xi)$, Eq. (3) is satisfied only if $P(x)\nu(x)=\text{const}$. Therefore we obtain either

$$P(x)=1/\nu(x) \tag{4}$$

or a singular solution. The letter emerges if $1/\nu(x)$ possesses a pole at some x. More precisely, if $\nu(x)$ has n zeros $x_1,\ldots,x_n$, the singular solution reads

$$P(x)=\sum_{i=1}^{n}a_i\delta(x-x_i)\Big/\sum_{i=1}^{n}a_i \tag{5}$$

for any set $\{a_i\}$. In contrast to the ordinary KP, $P(x)$ does not depend on Q. Finally, if $\nu(x)$ does not rise sufficiently fast with $|x|$, the stationary solution does not exist. It is the case, e.g., for $\nu(x)=\text{const}$.

To obtain the general solution of Eq. (2), we take the Laplace transform of $p(x,t)$ in respect to time: $G(x,s)=\mathcal{L}(p(x,t))$. The solution reads

$$G(x,s)=\int P(x,s|x')p_0(x')dx', \tag{6}$$

where $p_0(x)$ denotes the initial condition. The propagator $P(x,s|x')$ is the Laplace transform of the conditional probability of passing from x' to x during time t. It can be obtained analogously as for the KP [2] by taking into account all possible paths leading from x' to x and summing over jumps:

$$P(x,t|x') = \exp[-\nu(x')t]\delta(x-x') + P_1(x,t|x') + \sum_{k=2}^{\infty} \int P_k(x,x_1,\ldots,x_{k-1},t|x')dx_1\cdots dx_{k-1}. \tag{7}$$

The probability density P_k in the above expression corresponds to transition involving k jumps and can be obtained by a recurrence formula. Finally we have

$$G(x,s) = \frac{p_0(x)}{s+\nu(x)} + \frac{1}{s+\nu(x)} \int \frac{\nu(x_0)p_0(x_0)Q(x-x_0)}{s+\nu(x)}dx_0 + \frac{1}{s+\nu(x)} \sum_{k=2}^{\infty} \int \frac{\nu(x_0)p_0(x_0)}{s+\nu(x_0)} Q(x-x_{k-1}) \times \left[\prod_{i=2}^{k} \frac{\nu(x_{i-1})Q(x_{i-1}-x_{i-2})}{s+\nu(x_{i-1})}dx_{i-1}\right]dx_0. \tag{8}$$

Note that, in contrast to the ordinary KP, integrals cannot be factorized. It is clear from the construction that the kth term in Eq. (8) corresponds to a path with k jumps. If we are interested only in a solution for a small time or we choose the initial condition close to the stationary limit, the series can be cut at relatively small k. Otherwise, a Monte Carlo simulation is a better method to obtain the general solution [10].

A natural field of applications of the GKP is transport phenomena. Let us interpret the GKP value x as the one-dimensional particle velocity v, characterized by probability distribution $f(v,t)$. The particle changes its velocity according to the distribution $Q(v-v')$ and with the rate $\nu(v)$. Using Eq. (2), we can write down the evolution equation for $f(v,t)$:

$$\frac{\partial}{\partial t} f(v,t) = -\nu(v)f(v,t) + \int K(v,v')f(v',t)dv', \tag{9}$$

where $K(v,v')=Q(v-v')\nu(v')$. The thermal equilibrium with temperature T the system is supposed to reach asymptotically,

$$P(v) = f_e(v) = 1/\sqrt{2\pi T}\,\exp(-v^2/2T), \tag{10}$$

imposes the form for jumping rate $\nu(v)$, according to Eq. (4). Equation (9) has the same structure as the one-dimensional linear Boltzmann equation with the collision kernel $K(v,v')$. In the framework of GKP, we do not consider individual, deterministic collisions between gas particles; the velocity change is treated stochastically and governed by a given probability distribution. The kernel satisfies the detailed balance condition in equilibrium. Moreover, the collision rate $\nu(v)=\int K(v',v)dv'$ is variable. Due to the factorized form of the kernel K, the kinetic equation becomes easier to solve than for a general case (e.g., [11]). In particular, the collision rate completely determines the asymptotic distribution.

Figure 1 presents an example of the transport process modeled by GKP; Eq. (9) has been solved for the initial condition $p_0(v)=1/\sigma_0\sqrt{2\pi}\,\exp[-(v-1)^2/2\sigma_0^2]$, with $\sigma_0=0.1$, and $Q(\xi)=1/\sigma\sqrt{2\pi}\,\exp(-\xi^2/2\sigma^2)$, with $\sigma=1$. The

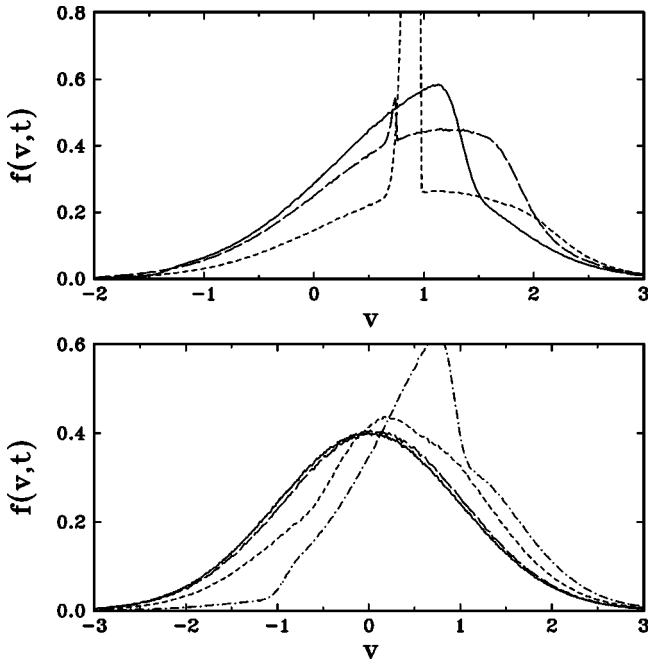

FIG. 1. Probability distribution $f(v,t)$ calculated for both $Q(\xi)$ and $p_0(v)$ in the form of Gaussians with parameters $\sigma=\sigma_0=1$. The thermal equilibrium has temperature $T=1$. Curves correspond to the following times. Upper part: $t=0.25$ (short-dashed line), $t=0.3$ (long-dashed line), and $t=0.4$ (solid line). Lower part: $t=0.5$ (dash-dotted line), $t=1$ (short-dashed line), $t=2$ (long-dashed line), and $t=3$ (solid line). The letter curve coincides with the asymptotic Maxwellian $f_e(v)$ [Eq. (10)].

temperature $T=1$. The picture can then be interpreted as relaxation of a cold gas approaching a hot environment. Distributions have been obtained by means of Monte Carlo simulations of individual trajectories; the statistical ensemble comprises 10^7 of them for each time. The figure indicates that the bias connected with initial value $v=1$ is kept for a long time, before the equilibrium distribution is finally reached. For the ordinary KP a memory about the initial condition can be preserved due to rather trivial reasons: only because there is a finite probability—proportional to $\exp[-\nu(v)t]$—that no jump occurs. In the GKP model, on the other hand, collisions between particles do not destroy memory completely. However, the speed of memory loss strongly depends on velocity. Since $\nu(v)\sim\exp(v^2)$, jumps are extremely frequent for large v. As a result, velocity values corresponding to infinitesimally small time intervals appear as statistically independent: we get the white noise limit. Moreover, despite the fact that velocity can be, in principle, arbitrarily large, only values around $v=0$ are effectively observed because only they persist for finite time intervals.

The probability density $Q(v-v')$ is important for the speed of the distribution tails relaxation towards equilibrium. We can expect that larger width of Q would result in faster equilibration because then large jumps are more probable. For example, if Q is the Gaussian characterized by the width σ, the size of the jump is proportional to σ: $v-v'=\xi_G \sim \pm\sigma\sqrt{-\ln r}$ where r is a random number uniformly distrib-

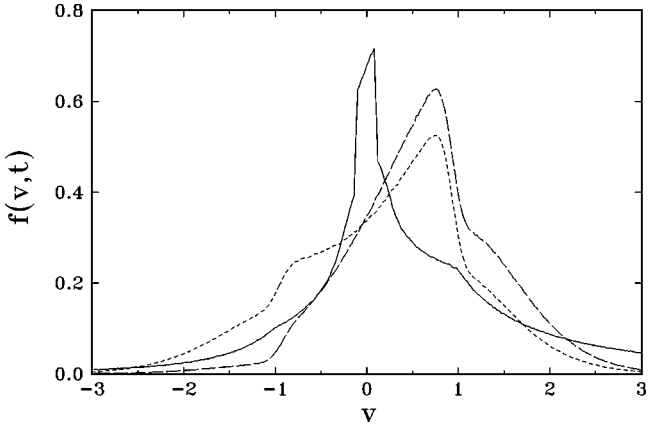

FIG. 2. Distribution $f(v,t)$ at $t=0.5$ calculated for $Q(\xi)$ in the form of Gaussian (long-dashed line) and Cauchy distribution (short-dashed line); both cases correspond to the Maxwellian equilibrium with $T=1$. The solid line corresponds to the algebraic form of the collision rate $\nu(v)$, Eq. (11), with $\alpha=0.5$ and $\beta=1.5$, as well as Gaussian Q.

uted in the interval $(0,1)$; the behavior of ξ_G near $r=0$ is responsible for large jumps. On the other hand, the random variable distributed according to the Cauchy distribution, $Q(\xi_C)=1/\pi(1+\xi_C^2)$, can be expressed as $\xi_C=1/\tan(\pi r/2)$. It rises to infinity faster than ξ_G does for any σ and variance becomes infinite: $\langle \xi_C^2 \rangle = \int_0^1 \xi_C^2\, dr = \infty$. Therefore, already a small number of jumps is able to spread velocity values over a region distant from the initial condition and to equilibrate the tails. Numerical calculations confirm a faster relaxation in the case of the Cauchy distribution; a comparison of the distributions for both kernels at $t=0.5$ is presented in Fig. 2.

A description of the transport processes in terms of kinetic equations, which is restricted to relaxation to the thermal equilibrium in the form of standard Maxwellian distribution, may not suffice. Recently, other forms of steady-state solutions of the Boltzmann equation have been discussed—e.g., the Lévy velocity distribution [12], characterized by an infinite second moment, with power-law tails. Such an overpopulation of the velocity distribution tails has physical reasons and may be caused by chemical reactions with a high activation energy [13]. An algebraic high-energy tail has been predicted [14] as a solution of the nonlinear Boltzmann equation for an inelastic freely cooling gas in the framework of the inelastic Maxwell model [15]. A similar tail has been found for the uniform shear flow [16]. Most of the above models of both linear and nonlinear Boltzmann equations assume a constant collision rate. For some physical systems, however, the variability of that rate cannot be neglected—e.g., if the transport mechanism is governed by long free paths. This concept is a foundation of a kinetic neutral particle transport model for long mean free path environments which has been recently applied to describe heat transfer in a rare gas between two parallel plates [17], emphasizing the importance of large Knudsen numbers. Similarly, it has been found [18] that long intervals of constant velocity may be responsible for the equilibration process in deep-inelastic, grazing collisions of atomic nuclei. The resulting energy distribution differs substantially from that for thermal equilibrium: it possesses a long tail and becomes infinite at small energy.

The GKP allows us to model easily any form of asymptotic distribution because it is fully determined by the jumping rate $\nu(v)$. Since the probability that no jump occurs during time t is given by $\exp[-\nu(v)t]$, the average interval of constant velocity v equals $1/\nu(v)$ and can be infinite. We can get steady-state solutions possessing such infinite intervals, as well as power-law tails, assuming, for example, the algebraic form of frequency:

$$\nu(v) = [v^\alpha \theta(1-|v|) + v^\beta \theta(|v|-1)]/N, \tag{11}$$

for $0<\alpha<1$ and $\beta>1$, where $N=[2(1/(1-\alpha))+1/(\beta-1)]^{-1}$ is the normalization constant. Figure 2 presents a comparison of the distribution for that frequency calculated at $t=0.5$ with the corresponding Maxwellian cases. Already at such short times it appears almost symmetric because it is attracted towards zero due to the abundance of long intervals of constant velocity. Nevertheless, full relaxation requires a similar time to that for the Maxwellian case.

Finally, let us consider the GKP for the case of constant frequency: $\nu(x)=\nu_0=\text{const}$, for which no equilibrium exists. The expression for the time-dependent probability distribution, given by Eq. (8), can be easily inverted, producing the following result:

$$p(x,t) = \exp(-\nu_0 t) p_0(x) + \exp(-\nu_0 t) \sum_{k=1}^{\infty} \frac{(\nu_0 t)^k}{k!} p_0(x) \star Q(x)^{(k-1)\star}, \tag{12}$$

where $\star$ denotes a convolution and $f^{k\star}$ is the k-fold convolution of the function f. If, in addition, we take $Q(\xi)$ in the Gaussian form $Q(\xi)=1/(\sigma\sqrt{2\pi})\exp(-\xi^2/2\sigma^2)$ and the initial condition as $p_0(x)=\delta(x)$, the k-fold integral can be evaluated:

$$p(x,t) = \exp(-\nu_0 t)\delta(x) + \exp(-\nu_0 t) \sum_{k=1}^{\infty} \frac{(\nu_0 t)^k}{\sigma\sqrt{2k\pi k!}} \exp[-x^2/(2k\sigma^2)]. \tag{13}$$

The width of the above distribution rises linearly with time. To demonstrate that this result is generic, let us calculate the characteristic function of $p(x,t)$: $\tilde{p}(k,t)$. Equation (2) is then of the form

$$\frac{\partial \tilde{p}(k,t)}{\partial t} = \nu_0 \tilde{p}(k,t)[\tilde{Q}(k)-1], \tag{14}$$

where $\tilde{Q}(k)$ stands for the characteristic function of $Q(\xi)$. The solution with initial condition $\tilde{p}_0(k)$ reads

$$\tilde{p}(k,t) = \tilde{p}_0(k) \exp\{\nu_0[\tilde{Q}(k)-1]t\}. \tag{15}$$

Differentiating Eq. (15) twice and putting $k=0$, we get the expression for the distribution width:

$$\langle x^2\rangle_p = \langle x^2\rangle_{p_0} + \nu_0(2\langle x\rangle_{p_0}\langle x\rangle_Q + \langle x^2\rangle_Q)t + \nu_0^2\langle x\rangle_Q^2 t^2. \tag{16}$$

The interpretation of Eq. (16) is straightforward. If the first moment of Q vanishes and variance of both Q and p_0 is finite, we get normal diffusion with the diffusion coefficient $\mathcal{D}=\nu_0\langle x^2\rangle_Q$. The nonvanishing average of Q, on the other hand, produces a bias which results in a strong enhancement of the diffusion process: the variance of $p(x,t)$ rises as t^2, and the diffusion is ballistic. Finally, if $\langle x^2\rangle_Q$ is divergent, as is the case, e.g., for the Cauchy distribution, the width of $p(x,t)$ becomes infinite for any time.

A peculiarity of the GKP for a constant jumping rate is that the covariance is the same as variance: $\mathcal{C}(t_0,\tau) = \langle x^2\rangle_{p(x,t_0)}$, where τ means the time increment and t_0 is the initial time; i.e., the covariance does not depend on τ but it does depend, linearly, on t_0. That statement—valid for any even $Q(\xi)$—can be proved by Laplace transforming of the following formula [19]:

$$\mathcal{C}(t_0,\tau) = \int\int x'(t_0)x(t_0+\tau)P(x,\tau\,|\,x')p(x',t_0)dxdx', \tag{17}$$

where the conditional probability $P(x,\tau|x')$ is given by Eq. (7); the integrals can be performed one after the other.

To conclude, the presented modification of the KP, which takes into account the size of subsequent jumps, is physically more realistic than the original KP. It can describe the transport phenomena: both kinetic processes, relaxing to the thermal equilibrium, and diffusion. A simple form of the master equation steady-state solution offers an easy way to handle problems characterized by long-time velocity distributions exhibiting deviations from the Maxwellian.

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